

Models for the evolution of free granular surfaces

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Abstract

We introduce two sets of continuum equations to describe granular flow on a free surface and study their properties. The equations derived from a microscopic picture that includes jumps and a mobility threshold, account for ripple and crater formation.

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1 Introduction

One of the apparently simple problems for which there is no consensus yet on which are the correct continuum equations of motion is the flux of grains on a granular surface (see for example ref. [1, 2, 3]). This flux can be driven by a fluid, like air, by gravity or by an initial impact. It modifies the granular surface itself either by deposition or by erosion. We will consider in this paper a gravity driven flow with variable initial energy and consider deposition only. In addition we want to be in the limit of small flux. One realization is the slow buildup of a pile from a point source.

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A discrete model was proposed in ref. [4] in which particles jump down stairs and deposit if their energy is below a certain material dependent threshold. From this model a set of continuum equations was derived in ref.[5] and applied to the problem of impact craters. This continuum description, however, made an assumption on the time dependence of the slope that only contained the threshold in an indirect way, namely by inserting the known steady state result. Also all the jumps were of length unity even if their kinetic energy was different. In this paper we want to deal with these defects of the equations in ref. [5] by inserting the two mentioned effects independently into the equations and investigating the consequences analytically and numerically.

2 Modelling jumps of grains

We will introduce an extension to the model proposed in ref. [5]. There, particles of mass m are added at the top of a pile, and they move, under the action of gravity g , until their kinetic energy e falls below a certain threshold U (related to the friction between grains) where they then stop. At each collision the kinetic energy decreases by a factor proportional to the restitution coefficient of the material r . Mathematically, it reads:

$$e(x + \delta) = (e(x) + mg(h(x) - h(x + \delta)))r \quad (1)$$

It was assumed in ref. [5] that the traveled distance after each particle jump is constant. Here, trying to model a more realistic situation, we assume that this distance is proportional to the kinetic energy of the particle. So, we propose to modify equation (1) to:

$$e(x + qe) = (e(x) + mg(h(x) - h(x + qe)))r \quad (2)$$

where q is a proportionality coefficient (in units of $\frac{1}{mg}$). Then, writing eq. (2) in differential form we obtain:

$$\frac{de}{dx} = \frac{1}{q}(r - 1) + r\gamma(x) \quad (3)$$

where $\gamma(x) = -mgdh/dx$.

To this equation one adds in ref.[5] an evolution equation for the slope of the form:

$$\frac{d\gamma}{dx} = \Gamma(U - e(x)) \quad (4)$$

where Γ controls the rate of relaxation of the slope. Boundary conditions were imposed trying to reproduce real experimental situations of heap formation: $e(0) = e_o$ a value which is proportional to the height from which the grains fall on the pile, and $\gamma(0) = 0$.

The equations (3) and (4), constitute a pair of linear differential equations with constant coefficients. From them, the angle of a pile can be easily calculated:

$$\gamma(x) = \frac{1-r}{rq}(1 - \cos(\sqrt{\Gamma r}x)) - \sqrt{\Gamma r}(e_o - U) \sin(\sqrt{\Gamma r}x) \quad (5)$$

and then, integrating equation (5) the shape of the pile is:

$$h(x) = h(0) - \frac{1-r}{rqmg}x + \frac{1-r}{rqmg} \frac{1}{\sqrt{\Gamma r}} \sin(\sqrt{\Gamma r}x) - \frac{1}{rmg}(e_o - U) \cos(\sqrt{\Gamma r}x) \quad (6)$$

This solution has two remarkable properties as shown in Figure 1. First of all the oscillatory behaviour superimposed to the usual linear decay of the pile. These oscillations arise from the existence of a length scale Γ which controls the rate of the relaxation of the slope while independently the particles jump a distance (qe) determined only by their kinetic energy. This phenomenon is similar to the formation of ripples due to saltation [6], and it could be the first theoretical prediction of “gravity” induced ripples.

Another important feature of eq. (6) is that it can reproduce a crater formation on the top of pile. In fact, in the limit $\sqrt{\Gamma r}x \ll 1$, equation (6) transforms into:

$$mgh(x) = mgh(0) - \frac{1}{r}(e_o - U)(1 - \frac{\Gamma r}{2}x^2) \quad (7)$$

showing the existence of a parabolic-like crater whose depth linearly increases with the initial kinetic energy of the particles.

3 Modelling the mobility threshold

In this model, we add to the equation for the energies from ref. [5]

$$\frac{de}{dx} = \frac{r-1}{\delta}e + r\gamma(x) \quad (8)$$

a new equation for the angles based on the assumption that the equilibrium local slope of the pile is constant, i.e.

$$\phi(x + \delta) = \phi(x) \quad (9)$$

where ϕ is the slope of the pile.

The evolution of the slope proceeds in the following way: if a particle arrives at x , it has two possibilities (as described for a similar model in ref. [4]), if $e > U$, it moves to $x + \delta$, then:

$$\phi(x + \delta) = \phi(x) - 1 \quad (10)$$

or, it stops when $e < U$ and

$$\phi(x + \delta) = \phi(x) + 1 \quad (11)$$

Equations (10) and (11) may be written in the following differential form:

$$\frac{d\phi}{dx} = \frac{1}{\delta} \text{sgn}(U - e) \quad (12)$$

We next solve the coupled differential equations (8) and (12) numerically. It is more convenient to write both equations in the same form. So dividing eq. (8) by mg , and noting that $\gamma = mg\phi$, we obtain the following equations:

$$\frac{dz}{dx} = \frac{r-1}{\delta}z + \frac{r}{\delta}\phi \quad (13)$$

$$\frac{d\phi}{dx} = \frac{1}{\delta} \text{sgn}(z_u - z) \quad (14)$$

where $mgz = e$ and $mgz_u = U$.

Then, once $\phi(x)$ is known the height of the pile can be calculated through integration of ϕ and obtained numerically.

To numerically solve the equations (13) and (14) the function $\text{sgn}(x)$ was replaced by the continuous function $\tanh(\alpha x)$. Profiles calculated using

different values of α are shown in Figure 2. From the Figure we conclude that if $\alpha > 10^2$ all the numerical results are equivalent. To be on the secure side we used in the rest of the paper $\alpha = 10^5$.

Figure 3 represents typical sandpile profiles obtained for different values of the parameters. The bottom curve (where the crater is better defined) represents a system where the grains have higher initial energy.

In figure 4 is plotted the equilibrium angle of the pile ϕ_{eq} as a function of $U(1-r)/r$ for different values of e_o . As previously calculated in ref. [4], the angle of the pile is equal to $U(1-r)/r$ showing the equivalence between our approach and that proposed in [5].

The depth Δh and the width Δx of the craters, should only depend on the three parameters involved in the model r, e_o and U . In fact, in figures 5 and 6, Δh and Δx obtained using different sets of parameters (r, e_o, U) collapse on the same curve following the scaling relations:

$$\Delta h = \left(\frac{e_o}{U} - 1\right)^\alpha f\left(\left(\frac{e_o}{U} - 1\right)^{-\nu} r\right) \quad (15)$$

and

$$\Delta x = \left(\frac{e_o}{U} - 1\right)^\beta g\left(\left(\frac{e_o}{U} - 1\right)^{-\nu} r\right) \quad (16)$$

where $\beta = 2.0$, $\alpha = 1.0$, $\nu = 4.0$ and $f(x)$ and $g(x)$ are scaling functions with the following properties $f(x) \sim 1$ if $x \ll 1$ and $f(x) \sim 0$ if $x \gg 1$, idem for g . This indicates that around $e_o = U$ and $r = 0$ we have critical behaviour with simple integer exponents.

Physically equations (15) and (16) imply that in granular systems with negligible restitution coefficient ($r \sim 0$) the depth of the crater increases linearly with $e_o/U - 1$ and with $1/r$, in good agreement with eq.7 while the width of the craters increases parabolically ($\beta = 2.0$). Therefore systems with small e_o , i.e. ($e_o \approx U$) or $r \approx 1$ do not develop a crater.

4 Conclusions

We have presented two sets of equations of motion for the limit of dilute granular surface flow. The first set included a jump length proportional to the energy of the grains. This gave ripple formation on the critical slope, a phenomenon which has not yet been observed experimentally. The second

set was a cleaner way to include the deposition threshold than done in ref. [5]. The results confirm that the previously used equation (introduced in ref. [5]) gave qualitatively the right answer for the shape of the pile which means that the picture of a characteristic length for relaxation to the equilibrium angle is correct.

Acknowledgments

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Figure Captions

Figure 1 Graphical representation of equation 6. We chose $\frac{1-r}{rq} = 0.1, \sqrt{\Gamma r} = 1$ and $\frac{e_o - U}{r} = 0.1$.

Figure 2 Shape of the pile calculated by integrating the solution ϕ of equations (13) and (14) for $\alpha = 1$ (upper curve) and $\alpha = 10$ and 100 (bottom curve).

Figure 3 Shape of the pile for $r = 0.03, z_U = 0.05$, and (from top to bottom) $z = 0.05, 0.10$ and 15 . The profiles are shifted in the z axis to distinguish them better.

Figure 4 Equilibrium angle of the pile ϕ_{eq} as a function of $U(1-r)/r$ for different values of e_o, r and U .

Figure 5 Data collapse of the depth of the craters using eq: (15). Parameters: $(r, e_o, U) = \triangle (0.01, 0.15, 0.05); \square (0.01, 0.25, 0.05); \odot (0.01, 0.25, 0.10); \bullet (0.01, 0.25, 0.15); \times (0.03, 0.20, 0.10); + (0.03, 0.25, 0.15); \square (0.05, 0.30, 0.10)$.

Figure 6 Data collapse of the depth of the crater using eq: (16). Parameters: $(r, e_o, U) = \triangle (0.01, 0.15, 0.05); \square (0.01, 0.25, 0.05); \odot (0.01, 0.25, 0.10); \bullet (0.01, 0.25, 0.15); \times (0.03, 0.20, 0.10); + (0.03, 0.25, 0.15); \square (0.05, 0.30, 0.10)$.











